MA 405 - 601 - Test #2

Friday, October 15, 2021

Instructions: *Please read all instructions carefully.*

- *You have* 100 *minutes to take the exam (*75 *for the actual work and* 25 *for uploading your solutions).*
- *Work each problem neatly on your own paper. You will need to scan your work and submit it on Gradescope. Keep the original copy of your work in case of any technical difficulties.*
- *You will need to indicate where each problem is, so* ***start each problem on a new page****.*
- *You may* ***not*** *use a calculator, your notes, the internet, or any other testing aid.*
- *You must show* ***all of your work*** *for credit. Partial credit is given for partial solutions.*

1. Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{n\times k}$, $C \in \mathcal{M}_{k\times m}$, and $D \in \mathcal{M}_{n\times m}$. Determine the dimensions of each of the following. If it is not defined, say so.

   (a) AB

   (b) $C^T B$

   (c) $B(DC^T)^T$

   (d) $(A^T C^T)^T B$

   (e) $(A - D^T)B$

   (f) $BC - A$

2. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -4 \\ 1 \\ 1 \end{bmatrix}.$$

   (a) Explain why S is not a basis for $\mathbb{R}^4$.

   (b) Show that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

   (c) Find $\dim(\text{span}(S))$.

   (d) Find a basis for $\mathbb{R}^4$ that contains $\mathbf{v}_1$ and $\mathbf{v}_2$.

3. Consider the linear system below:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ \quad\;\; - 4x_2 \;\; + x_3 = -1 \\ \quad\;\;\;\; 3x_2 \;\; - x_3 = 1 \end{cases}$$

   (a) Write the system as a matrix equation (i.e. $A\mathbf{x} = \mathbf{b}$).

   (b) Find A^{-1}.

   (c) Use your answer from part (b) to solve for $\mathbf{x}$.

4. Let $A \in \mathcal{M}_{2\times 4}(\mathbb{R})$.

   (a) If $\text{null}(A)$ is a subspace of $\mathbb{R}^p$, what is p?

   (b) If $\text{col}(A)$ is a subspace of $\mathbb{R}^k$, what is k?

   (c) What is the nullity of A if $\text{rank}(A) = 1$?

5. Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^{2\times 2}$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y & 0 \\ x + y - z & 3y + z \end{bmatrix}$.

   (a) Prove that T is a linear transformation.

   (b) Find a basis for $\text{null}(T)$ and determine $\text{nullity}(T)$.

   (c) Find a basis for $\text{range}(T)$ and determine $\dim(\text{range}(T))$.

6. Mark each statement as True or False. Justify.

   (a) If A is invertible, then $\text{null}(A) = \{\mathbf{0}\}$.

   (b) If $A \in \mathbb{R}^{4\times 7}$, and $\text{nullity}(A) = 3$, then for any $\mathbf{b} \in \mathbb{R}^4$, $A\mathbf{x} = \mathbf{b}$ has at least one solution.

   (c) If $CD = \mathbf{0}_{n\times n}$, then $C = \mathbf{0}$ or $D = \mathbf{0}$.

   (d) For $B \in \mathbb{R}^{p\times k}$, $\text{rank}(B) + \text{nullity}(B) = p$.

   (e) $T : \mathcal{M}_{2\times 2} \to \mathcal{P}_2$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (c + d)x^2 + (a^2 - a)x + (a + b + c - d)$ is a linear transformation.

7. Choose **one** of the following to complete (7A or 7B), and clearly indicate which problem you are completing.

   (7A) Suppose $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $c_1\mathbf{v}_1, +c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = 0$ with $c_1 \neq 0$. Show that $V = \text{span}\{\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$.

   (7B) Suppose that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is an invertible matrix. Show that $\mathcal{B}' = \{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n\}$ is also a basis for $\mathbb{R}^n$.

8. (a) Write the following statement, and sign your name: *I affirm that I have neither given nor received unauthorized aid on this test.*

   (b) Scan and upload your work to Gradescope before the deadline. Make sure that you select what page each question is on.

9. **Bonus** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ 0 \end{bmatrix}$$

   and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 x_2 \end{bmatrix}$$

   (a) Is ST a linear transformation? Justify.

   (b) Is TS a linear transformation? Justify.

   (c) True or False: $ST(\mathbf{v}) = (I \circ T)(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^2$, where I is the identity operator.

   (d) True or False: $\text{range}(T) = \mathbb{R}^2$.